Complex Co-Evolutionary Dynamics – Structural Stability and Finite Population Effects

Peter Tiňo, Siang Yew Chong, and Xin Yao

Abstract—**Unlike evolutionary dynamics, co-evolutionary dynamics can exhibit a wide variety of complex regimes. This has been confirmed by numerical studies e.g. in the context of Evolutionary Game Theory (EGT) and population dynamics of simple two-strategy games with various types of replication and selection mechanisms. Using the framework of shadowing lemma we study to what degree can such infinite population dynamics (1) be reliably simulated on finite precision computers and (2) be trusted to represent co-evolutionary dynamics of possibly very large, but finite populations. In a simple EGT setting of two-player symmetric games with two pure strategies and a polymorphic equilibrium we prove that for (μ, λ), truncation, sequential tournament, best-of-group tournament and linear ranking selections, the co-evolutionary dynamics do not possess the shadowing property. In other words, infinite population simulations cannot be guaranteed to represent real trajectories or to be representative of co-evolutionary dynamics of potentially very large, but finite populations.**

I. INTRODUCTION

In this study we concentrate on a class of evolutionary algorithms (EAs) known as co-evolutionary algorithms (CEAs). Unlike classical EAs that require an *absolute* quality measurement of solutions to guide the population-based stochastic search process, in CEAs the solution quality can be estimated only with respect to its performance against a (usually) small sample of test cases (e.g. members of a co-evolving (sub-)population) [1]. In cases where an absolute quality measurement is not available, CEAs can still solve the problem by making use of some form of *strategic* interactions between competing solutions in the population to guide the search [2]. A significant body of work on co-evolution has been devoted to the development of CEAs in the context of game playing (see e.g. [3]).

Despite early success of CEAs (e.g. in solving games), there have been well-documented failures leading to poor performance of CEAs under certain conditions. One example is the *overspecialization* of evolved game strategies that specialize to and perform well only against specific type of opponents, rather than being able to compete against a wide range of opponent types [2], [4]. Such effects can result in oscillatory behavior or much more complex dynamical regimes (even chaos [5]) that are not to be seen in the context of classical EAs employing an absolute fitness measure. Most studies of complex co-evolutionary dynamics are usually (and conveniently) performed under the assumption of infinite populations, where the entities of interest are, e.g., the ratios of individuals adopting a particular strategy.

Dynamical maps [5] were used for a simple two-strategy game and various types of replication and selection pressure mechanisms. Dynamics of the maps were then studied in a series of numerical simulations. Under a wide range of parameter settings[1], the population dynamics showed clear

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[1] Payoff parameters, as well as selection and replication parameters.

signs of chaotic behavior. While such studies are very valuable for true appreciation of dynamical intricacies that can be associated with co-evolutionary dynamics, it has never been shown that complex dynamics represented through numerically generated trajectories can actually represent or even approximate theoretically true co-evolutionary dynamics. In chaotic dynamics, nearby trajectories get locally exponentially separated and so round-off errors of computer arithmetic will lead to numerical trajectories very different from the 'real' ones described by the infinite population equations. Unless one uses methods of constructive mathematics to generate trajectories to arbitrary precision [6], one can legitimately ask how informative the observed 'chaotic' trajectories are, given that the computer precision is limited. There is a more subtle issue associated with this research question: even in the case of infinite populations, the population ratios cannot be irrational. Yet, there is no guarantee that the images of population ratios under the non-linear dynamical map will be rational. This does not lead to major complications when EAs with relatively simple dynamics are considered. However, in the case of complex co-evolutionary dynamics the situation is much less clear.

Even more fundamental is the question whether the theoretically described or numerically observed dynamical intricacies of infinite population models have any relation to the dynamics of potentially very large, but finite populations. In other words, how informative are the infinite population studies about the finite population 'practice' and is there a fundamental limit that hampers applicability of infinite population studies to finite population scenarios (even in the case of 'very large' populations)? The effects of finite populations on replicator dynamics in two-strategy games with polymorphic fitness-equilibrium (between the two pure strategies) were studied in [7]. However, the framework there is completely different from that adopted in this study. While we consider replicator dynamics as deterministic processes operating on infinite (or very large) populations of pure-strategists, the previous work [7] studied *stochastic* replicator dynamics operating on possibly small finite pure-strategies' populations.

In this study we propose to address all the issues raised previously in the context of structural stability of dynamical systems. In particular, we will use the framework of shadowing lemma developed for certain classes of chaotic dynamical systems (see e.g. [8]). For brevity we will consider population dynamics under co-evolution in games with two pure strategies. However, the framework of η-isolating fixed point of the population dynamics (introduced in this study) can be used in more general settings, e.g. games with more than two pure strategies.

The paper has the following organization: In section II we introduce the concepts of pseudo-trajectories and their shadowing by true trajectories. We also prove a result about the lack of shadowing in a class of dynamical systems related to population dynamics considered in this paper. Section III briefly introduces EGT and studies discontinuity at equilibrium for several types of selection mechanisms. In section IV we state the main results concerning shadowing of infinite population replication dynamics. The main findings are discussed and summarized in section V.

II. THE SHADOWING PROPERTY

Consider a discrete time dynamical system $F : X \to X$ on a metric space (X, d). Given an initial condition $x_0 \in X$, the map F generates an orbit $x_n = F(x_{n-1}) \in X$, $n = 1, 2, ...$. If instead of the true iterands x_n we observed x_n corrupted by a bounded noise, but still used the dynamics F, we would obtain a *pseudo-trajectory* $\{\tilde{x}_n\}_{n\geq 0}$,

$$d(\tilde{x}_0, x_0) < \delta, \quad d(F(\tilde{x}_{n-1}), \tilde{x}_n) < \delta, \quad n \geq 1,$$

where $\delta > 0$ is the range of the bounded noise. Such a pseudo-trajectory is often referred to a δ*-pseudo-trajectory* [9].

Given an $\epsilon > 0$, we say that a trajectory $\{y_n\}_{n\geq 0}$ ϵ-shadows another trajectory $\{x_n\}_{n\geq 0}$, if $\{y_n\}_{n\geq 0}$ stays within the ϵ-tube around $\{x_n\}_{n\geq 0}$:

$$d(x_n, y_n) < \epsilon, \quad n \geq 0.$$

The Shadowing lemma (e.g. [8]) tells us that (remarkably) even for the most complex and locally exploding chaotic maps, under some circumstances, the corrupted pseudo-trajectories are informative: For *any* $\epsilon > 0$, there exists a $\delta > 0$, such that for *every* δ-pseudo-trajectory $\{\tilde{x}_n\}_{n\geq 0}$ there is a true (uncorrupted) trajectory $\{y_n\}_{n\geq 0}$ under F that ϵ-shadows the pseudo-trajectory $\{\tilde{x}_n\}_{n\geq 0}$:

$$d(\tilde{x}_n, y_n) < \epsilon, \quad y_{n+1} = F(y_n), \quad n \geq 0.$$

Hence, even though one could be tempted to assume that, under chaos, trajectories that are disrupted at every point by a bounded noise cannot possibly represent anything real, in fact such trajectories can be closely shadowed by true trajectories.

Under what conditions can the shadowing property be guaranteed? Virtually all studies of the shadowing property have been performed in the framework of continuous and smooth dynamical systems. If a system is (uniformly) hyperbolic on an invariant set[2], then the system will have the shadowing property [10]. However, in general, establishing that a dynamical system is hyperbolic can be rather difficult. For discontinuous systems, the shadowing property must be more-or-less established on a case-by-case basis.

The key idea of this paper is illustrated in figure 1. Consider (a possibly complex) infinite-population dynamics F. An infinite population state x_n at time step n gets mapped to the infinite population state $x_{n+1} = F(x_n)$. If instead of x_{n+1} we used its 'corrupted' version $\tilde{x}_{n+1}$ - e.g. x_{n+1} corrupted by finite computer precision, or state of a large, but finite sub-population - to which the true dynamics F is applied, we would obtain a pseudo-trajectory of F. The key question is whether such pseudo-trajectories can possibly represent anything about the real infinite population dynamics F. In other words, whether they can be shadowed by true F-trajectories $\{y_n\}_{n\geq 0}$.

We will now show, in a rather general setting, that a certain class of (discrete time) dynamical systems particularly relevant to the co-evolutionary population dynamics considered in this paper does not possess the shadowing property. Recall that for $x \in X$ and $\epsilon > 0$, ϵ-neighborhood of x is an open set $U \subseteq X$ of diameter[3] ϵ containing x. The subsets of X are defined as open according to the topology induced on X by the metric d. If U is an ϵ-neighborhood of $x \in X$, then the neighborhood U with x taken out will be denoted by U_{-x}, i.e. $U_{-x} = U \setminus \{x\}$.

Definition 1: Consider a metric space (X, d) and a map $F : X \to X$. For a given $\eta > 0$ we say that a point $q \in X$ is an *η-isolating point of discontinuity of F*, if there exists $\rho > 0$ such that for every ϵ-neighborhood $U \subset X$ of q, $0 < \epsilon < \rho$, it holds[4]

$$F(U_{-q}) \cap B_\eta(F(q)) = \emptyset,$$

where

$$B_\eta(y) = \{x \in X \mid d(x, y) < \eta\}$$

is the open ball of radius η around $y \in X$.

The next theorem not only tells us that dynamical systems with discontinuities at fixed points cannot have shadowing property but also quantifies "to what degree" the shadowing cannot be guaranteed in such systems.

Theorem 2: Consider a discrete time dynamical system $F : X \to X$ on a metric space (X, d) with a fixed point $q = F(q) \in X$. If for some $\eta > 0$, q is an η-isolating point of discontinuity of F, then the dynamics F does not possess the shadowing property. In particular, no ϵ-shadowing is possible for $0 < \epsilon < \eta$ in the sense that for each $0 < \epsilon < \eta$ no $\delta > 0$ can be found so that every δ-pseudo-trajectory would be ϵ-shadowed by a true trajectory under F.

Proof: For some $\delta > 0$, consider a δ-pseudo-trajectory .... $\tilde{x}_{n-2}$ $\tilde{x}_{n-1}$ $\tilde{x}_n$ of F that gets within δ-neighborhood of q, i.e. $d(F(\tilde{x}_n), q) < \delta$. We can then set $\tilde{x}_{n+1} = q$ and since q is a fixed point of F, the pseudo-trajectory can stay in q for

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[2] Loosely speaking, at each point x of the invariant set the (linearized) system has only local contracting and expanding subspaces that get consistently translated by F into the local contracting and expanding subspaces at $F(x)$.

[3] Supremum of pairwise distances (under d) of points from U.

[4] The domain of F can be extended to subsets of X in the obvious manner: for any $B \subseteq X$, $F(B) = \{F(x) | x \in B\}$.

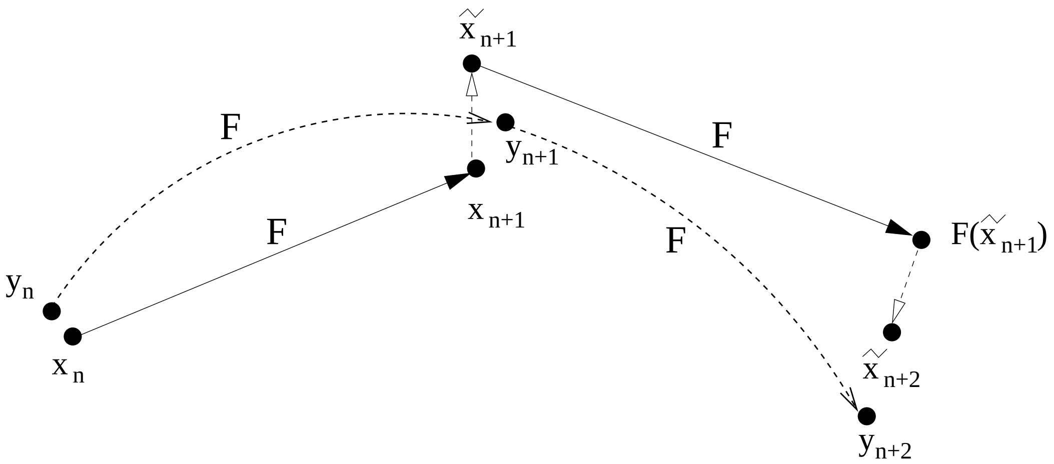

Fig. 1. Illustration of pseudo-trajectory $\{\tilde{x}_n\}_n$ of F, shadowed by the actual F-trajectory $\{y_n\}_n$.

an *arbitrary* number m of time steps,

$$\tilde{x}_{n+1} = \tilde{x}_{n+2} = ... = \tilde{x}_{n+m} = q.$$

Since q is both a fixed point of F (i.e. $q = F(q)$) and an η-isolating point of discontinuity of F, there exists $\rho > 0$ such that for every ϵ-neighborhood U of q, $0 < \epsilon < \rho$, we have $F(U_{-q}) \cap B_\eta(q) = \emptyset$. Denote the sphere of radius $\nu > 0$ around $y \in X$ by $S_\nu(y)$, i.e.

$$S_\nu(y) = \{x \in X \mid d(x, y) = \nu\}.$$

Having stayed in q for m time steps, we let the next element of the δ-pseudo-trajectory be

$$\begin{aligned} \tilde{x}_{n+m+1} &\in S_\nu(F(q)) \\ &= S_\nu(q), \end{aligned} \tag{1}$$

for some $0 < \nu < \min\{\rho, \delta\}$. That implies $d(F(\tilde{x}_{n+m+1}), q) \geq \eta$.

Now, set $\tilde{x}_{n+m+2} = F(\tilde{x}_{n+m+1})$ and fix ϵ such that $0 < \epsilon < \eta$. Recall that for shadowing property to hold we would need to have that for any $\epsilon > 0$, there is a $\delta > 0$, such that every δ-pseudo-trajectory can be ϵ-shadowed by an F-trajectory. However, the pseudo-trajectory

$$\begin{aligned} ... \;\; & \tilde{x}_{n-2},\ \tilde{x}_{n-1},\ \tilde{x}_n,\ \tilde{x}_{n+1} = q,\ \tilde{x}_{n+2} = q,\ ..., \\ & \tilde{x}_{n+m} = q,\ \tilde{x}_{n+m+1} \in S_\nu(q), \\ & \tilde{x}_{n+m+2} = F(\tilde{x}_{n+m+1}) \end{aligned}$$

cannot be ϵ-shadowed by a true trajectory under F. For ϵ-shadowing we would need F-trajectories to be able to stay ϵ-close to q, for arbitrarily small $\epsilon > 0$ and for arbitrary number m of time steps. After that the F-trajectories would need to jump at least η-far from q. But this is not possible, because since q is an η-isolating point of discontinuity of F, the only way for a true trajectory to stay for arbitrary number of time steps ϵ-close to q is to stay exactly in q, from which there is no escape under F. Hence, no ϵ-shadowing by a true trajectory under F is possible for $0 < \epsilon < \eta$. ∎

We are now ready to discuss the kinds of co-evolutionary population dynamics to which our results can be applied.

III. EVOLUTIONARY GAME THEORY AND ISOLATING DISCONTINUITIES OF REPLICATION DYNAMICS

EGT [11] provides a natural testbed in which CEAs can be analyzed. In classical game theory a rational individual (player) has to choose between distinct strategies - the one that maximizes its payoff when interacting against another player, who in turn, also maximizes its own payoff. In contrast, the EGT setting involves an infinitely large population of players that are allowed to use a set of predefined strategies. These strategies are *inheritable* and all players compete for payoffs that decide their average reproductive success [12]. Different constructions (e.g., different games, different selection mechanisms etc.) will lead to different frequencydependent population dynamics [13]. As such, EGT provides a framework in which one can study the conditions that affect the success of some strategies over others in the population under evolutionary process.

There have been few studies that employed EGT to analyze CEAs. For example, a simple EGT setting of the hawk-dove game that involves interactions between two strategies has been used to investigate the evolutionary process of CEAs under various conditions - Fogel et al. [14] investigated the impact of finite population, while the study in [5] investigated the impact of selection mechanisms.

Standard EGT framework is based on several assumptions. First, populations have infinitely many players (agents), each of which has a finite set of *pure strategies* to choose from in every round of the game. Second, every player interacts with all the other players in the population (*complete mixing*). Each player accumulates payoff depending on the outcome of the games. Third, players reproduce in proportion to their cumulative payoffs. Reproduction is asexual and without variation, i.e., players generate clones as their offspring.

As in [5], we consider a simple EGT setting with a two-player game. Each player has a finite set of pure strategies to choose from. For brevity, we concentrate on *symmetric games* where the set of pure strategies is the same for every player. When a player chooses strategy i while the opponent chooses strategy j, the payoff (game outcome) for the first player is denoted by[5] g_{ij}.

For two *pure strategies* s_1, s_2 the possible payoffs for the first player (row) playing against the opponent (column) can be represented as a 2×2 payoff matrix

$$\begin{array}{c|cc} & s_1 & s_2 \\ \hline s_1 & a & b \\ s_2 & c & d \end{array} \tag{2}$$

where each entry gives the respective payoff for the chosen pair of strategies. For example, the first player receives the payoff b when it chooses strategy s_1 while its opponent chooses s_2.

As already mentioned, each player in the population chooses only one of the two pure strategies. Assume that fraction $p \in [0,1]$ of players in the population chose to play s_1. Then $1-p$ is the proportion of players in the population that played s_2. The cumulative payoffs w_{s_1} and w_{s_2} for pure strategies s_1 and s_2, respectively, are given by

$$\begin{aligned} w_{s_1} &= ap + b(1-p) \\ w_{s_2} &= cp + d(1-p). \end{aligned} \tag{3}$$

To ensure existence of a population state in which the cumulative scores for both strategies are the same ($w_{s_1} = w_{s_2}$), we constrain the payoff structure so that $a < c$ and $b > d$ [5]. Such ‘equilibrium’ state is know as *polymorphic equilibrium*. In our case for each allowed payoff setting there is a unique polymorphic equilibrium

$$p_{EQ} = \frac{d-b}{a-c+d-b}. \tag{4}$$

Interpreting the population as a mixed strategy s (i.e. use pure strategies s_1 and s_2 with probability p and $1-p$, respectively), the state p_{EQ} is a *Nash equilibrium*, whereby the mixed strategy s is its own *best reply*. If a player uses s, the opponent obtains highest payoff when using s as well. Hence, in Nash equilibrium neither of the two players has an incentive to deviate unilaterally to use a different strategy.

An example of a game that satisfies the constraints $a < c$, $b > d$, is the classical game setting of the *hawk-dove game* which involves interactions of two distinct behaviors (pure strategies), *hawk* and *dove*, competing for gains G upon winning under the costs C of injury. Hawks are aggressive and two hawks will fight until one retreats with an injury. Interactions between hawks lead to the expected payoff $(G-C)/2$, given a probability of $1/2$ for injury. Two doves, in contrast, will avoid a fight and perform threatening postures until both retreats without injury. In such a case, they share the gain $G/2$. Any interaction between a hawk and a dove will lead to the dove retreating immediately. The hawk will take the full gain G, while the dove has zero gain, with no cost on injury incurred to both parties. The payoff matrix reads:

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[5] Payoff for the second player is then g_{ji}.

$$\begin{array}{c|cc} & \text{Hawk} & \text{Dove} \\ \hline \text{Hawk} & (G-C)/2 & G \\ \text{Dove} & 0 & G/2 \end{array} \tag{5}$$

When the cost of injury is greater than the gain in winning the game, $G < C$, the constraints $a < c, b > d$ are satisfied and the game has a unique polymorphic equilibrium $p_{EQ} = G/C$.

In EGT the population dynamics is described by the *replicator equation* that governs how the frequency/proportion of strategies in the population changes in the course of evolutionary process. Replicator equation under a selection mechanism based on the proportion of cumulative payoffs is given by

$$f(p) = \frac{p \cdot w_{s_1}}{p \cdot w_{s_1} + (1-p) \cdot w_{s_2}},$$

where $f(p)$ is the frequency of strategy s_1 in the population in the following generation ($t+1$), given that its frequency in the current population (generation t) is p.

In addition to such (classical) fitness-proportional selection, a variety of alternative selection mechanisms have been proposed. In the following we briefly introduce different selection mechanisms considered in this paper.

A. (μ, λ)-*selection*

The (μ, λ)-selection is usually associated with the selection operator used in a class of EAs known as *evolution strategies*. Each of μ parents generates k offspring, which results in $\lambda = k\mu$ offspring. In the case of infinite populations, we are only concerned with the parents-to-offspring ratio $\gamma = \mu/\lambda = 1/k$. The replicator equation has the form [5]:

$$f(p) = \begin{cases} 1 & \text{if} \quad p < p_{EQ} \text{ and } p \geq \gamma, \\ p/\gamma & \text{if} \quad p < p_{EQ} \text{ and } p < \gamma, \\ 1 + (p-1)/\gamma & \text{if} \quad p > p_{EQ} \text{ and } p > 1-\gamma, \\ 0 & \text{if} \quad p > p_{EQ} \text{ and } p \leq 1-\gamma, \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{6}$$

Note that when $k = 1$, we have a trivial case with replicator function f being the identity function. For larger k we have the following result for non-trivial equilibrium points $p_{EQ} \in (0,1)$:

Lemma 3: Consider the simple EGT setting of section III and a two-player symmetric game with two pure strategies and a polymorphic equilibrium $p_{EQ} \in (0,1)$. The population dynamics is driven by (μ, λ)-selection. Let $\kappa = \min\{p_{EQ}, 1 - p_{EQ}\}$. Then,

1) if $\lambda/\mu = 2$, for any sufficiently small $\rho > 0$, p_{EQ} is a $(\kappa - \rho)$-isolating point of discontinuity of f;
2) if $\lambda/\mu \geq 3$, p_{EQ} is a κ-isolating point of discontinuity of f.

Proof:

1) Let $k = \lambda/\mu = 2$. Then $\gamma = 1 - \gamma = 1/2$. We consider three distinct cases for the value of $p_{EQ} \in (0,1)$ with respect to $\gamma = 1/2$.
   - $p_{EQ} < \gamma$. In this case we have in the neighborhood of p_{EQ}:

     $$f(p) = \begin{cases} f_-(p) = p/\gamma = 2p & \text{if} \quad p \in [0, p_{EQ}), \\ f_+(p) = 0 & \text{if} \quad p \in (p_{EQ}, \gamma], \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{7}$$

     From the left, $\lim_{p \to p_{EQ}^-} f_-(p) = 2p_{EQ}$; from the right, $\lim_{p \to p_{EQ}^+} f_+(p) = 0$. Hence, for any sufficiently small $\rho > 0$, p_{EQ} is a $(p_{EQ} - \rho)$-isolating point of discontinuity of f.
   - When $p_{EQ} > \gamma$, we have in the neighborhood of p_{EQ}:

     $$f(p) = \begin{cases} f_-(p) = 1 & \text{if} \quad p \in [\gamma, p_{EQ}), \\ f_+(p) = 1 + \frac{p-1}{\gamma} & \\ = 2p - 1 & \text{if} \quad p \in (p_{EQ}, 1], \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{8}$$

     Hence, $\lim_{p \to p_{EQ}^-} f_-(p) = 1$ with separation from p_{EQ} equal to $(1 - p_{EQ})$ and $\lim_{p \to p_{EQ}^+} f_+(p) = 1 - 2(1 - p_{EQ})$ with the same separation from p_{EQ}. It follows that, for any sufficiently small $\rho > 0$, p_{EQ} is a $(1 - p_{EQ} - \rho)$-isolating point of discontinuity of f.
   - $p_{EQ} = \gamma$. Now we have in the neighborhood of

p_{EQ}:

$$f(p) = \begin{cases} f_-(p) = p/\gamma = 2p & \text{if} \quad p \in [0, p_{EQ}), \\ f_+(p) = 1 + \frac{p-1}{\gamma} & \\ = 2p - 1 & \text{if} \quad p \in (p_{EQ}, 1], \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{9}$$

It follows that $\lim_{p \to p_{EQ}^-} f_-(p) = 1$ with separation from p_{EQ} equal to $p_{EQ} = 1/2$ and $\lim_{p \to p_{EQ}^+} f_+(p) = 0$ with the same separation from p_{EQ}. Hence, for any sufficiently small $\rho > 0$, p_{EQ} is a $(p_{EQ} - \rho)$-isolating point of discontinuity of f.

2) The case $k = \lambda/\mu \geq 3$ is a bit more involved but again can be dealt with easily on a case-by-case basis. The interval $(0, 1)$ of possible positions of p_{EQ} is now split into three subintervals with $0 < \gamma < 1 - \gamma < 1$.

   - $p_{EQ} < \gamma$. In this case we have in the neighborhood of p_{EQ}:

$$f(p) = \begin{cases} f_-(p) = p/\gamma & \\ = kp & \text{if} \quad p \in [0, p_{EQ}), \\ f_+(p) = 0 & \text{if} \quad p \in (p_{EQ}, 1 - \gamma], \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{10}$$

   Now, $\lim_{p \to p_{EQ}^-} f_-(p) = kp_{EQ}$ with separation from p_{EQ} equal to $(k-1)p_{EQ}$. On the other hand, $\lim_{p \to p_{EQ}^+} f_+(p) = 0$, which is separated from p_{EQ} by p_{EQ}. Since $k \geq 3$, we have that p_{EQ} is a p_{EQ}-isolating point of discontinuity of f. The same analysis holds for the case $p_{EQ} = \gamma$.

   - When $\gamma < p_{EQ} < 1 - \gamma$, we have in the neighborhood of p_{EQ}:

$$f(p) = \begin{cases} f_-(p) = 1 & \text{if} \quad p \in [\gamma, p_{EQ}), \\ f_+(p) = 0 & \text{if} \quad p \in (p_{EQ}, 1 - \gamma], \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{11}$$

   The separation of $\lim_{p \to p_{EQ}^-} f_-(p) = 1$ and $\lim_{p \to p_{EQ}^+} f_+(p) = 0$ from p_{EQ} is $(1 - p_{EQ})$ and p_{EQ}, respectively. Setting $\kappa = \min\{p_{EQ}, 1 - p_{EQ}\}$, we can claim that p_{EQ} is a κ-isolating point of discontinuity of f.

   - $p_{EQ} = 1 - \gamma$. In the neighborhood of p_{EQ}:

$$f(p) = \begin{cases} f_-(p) = 1 & \text{if} \quad p \in [\gamma, p_{EQ}), \\ f_+(p) = & \\ 1 - k(1 - p) & \text{if} \quad p \in (p_{EQ}, 1], \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{12}$$

   The separation of $\lim_{p \to p_{EQ}^-} f_-(p) = 1$ from p_{EQ} is $1 - p_{EQ} = \gamma = 1/k$. Note that the separation of $\lim_{p \to p_{EQ}^+} f_+(p) = 0$ from p_{EQ} is greater, namely $p_{EQ} = 1 - 1/k$. Hence, p_{EQ} is a $(1 - p_{EQ})$-isolating point of discontinuity of f. The same analysis holds for the case $p_{EQ} = (1 - \gamma, 1)$.

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B. *Truncation Selection*

In truncation selection individuals below some quality threshold[6] are not allowed to create offspring. Truncation is often used in *evolutionary programming* [15]. The particular form of truncation selection analyzed in [5] has the following form: 1) The population is sorted according to the agents' evaluation scores; 2) the worst γ-fraction of the population is removed, $\gamma \in (0, 1/2)$, and 3) the removed agents are replaced with clones of individuals (no variation operators) in the best γ-fraction of the population. The replicator equation has the form [5]:

$$f(p) = \begin{cases} 1 & \text{if} \quad p < p_{EQ} \text{ and } p \geq 1 - \gamma, \\ p + \gamma & \text{if} \quad p < p_{EQ} \text{ and } p \in [\gamma, 1 - \gamma), \\ 2p & \text{if} \quad p < p_{EQ} \text{ and } p < \gamma, \\ p - \gamma & \text{if} \quad p > p_{EQ} \text{ and } p \in (\gamma, 1 - \gamma], \\ 2p - 1 & \text{if} \quad p > p_{EQ} \text{ and } p > 1 - \gamma, \\ 0 & \text{if} \quad p > p_{EQ} \text{ and } p \leq \gamma, \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{13}$$

In the context of two-strategy games, the replicator equation (13) (and hence our analysis here) is the same for the sequential tournament selection [5]. Discontinuity properties of the replicator map (13) are described in the following lemma.

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[6] The threshold can be determined by an absolute or relative quality measurement.

Lemma 4: Consider the simple EGT setting of section III and a two-player symmetric game with two pure strategies and a polymorphic equilibrium $p_{EQ} \in (0, 1)$. The population dynamics is driven by the truncation selection with parameter $\gamma \in (0, 1/2)$. Let $\kappa = \min\{p_{EQ}, 1 - p_{EQ}, \gamma\}$. Then, for any sufficiently small $\rho > 0$, p_{EQ} is a $(\kappa - \rho)$-isolating point of discontinuity of f.

Proof: We again proceed on a case-by-case basis, with the interval $(0, 1)$ of possible positions of p_{EQ} split into three subintervals deliminated by $0 < \gamma < 1 - \gamma < 1$.

- $p_{EQ} < \gamma$. In the neighborhood of p_{EQ}:
$$f(p) = \begin{cases} f_-(p) = 2p & \text{if} \quad p \in [0, p_{EQ}), \\ f_+(p) = 0 & \text{if} \quad p \in (p_{EQ}, \gamma], \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{14}$$
We have $\lim_{p \to p_{EQ}^-} f_-(p) = 2p_{EQ}$ with separation from p_{EQ} equal to p_{EQ}. Moreover, $\lim_{p \to p_{EQ}^+} f_+(p) = 0$, which is separated from p_{EQ} by p_{EQ}.
- $p_{EQ} = \gamma$. In the neighborhood of p_{EQ}:
$$f(p) = \begin{cases} f_-(p) = 2p & \text{if} \quad p \in [0, p_{EQ}), \\ f_+(p) = p - \gamma & \text{if} \quad p \in (p_{EQ}, 1 - \gamma], \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{15}$$
Now $\lim_{p \to p_{EQ}^-} f_-(p) = 2p_{EQ}$ and $\lim_{p \to p_{EQ}^+} f_+(p) = 0$.
- When $\gamma < p_{EQ} < 1 - \gamma$, we have in the neighborhood of p_{EQ}:
$$f(p) = \begin{cases} f_-(p) = p + \gamma & \text{if} \quad p \in [\gamma, p_{EQ}), \\ f_+(p) = p - \gamma & \text{if} \quad p \in (p_{EQ}, 1 - \gamma], \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{16}$$
The separation of both $\lim_{p \to p_{EQ}^-} f_-(p) = p_{EQ} + \gamma$ and $\lim_{p \to p_{EQ}^+} f_+(p) = p_{EQ} - \gamma$ from p_{EQ} is γ.
- $p_{EQ} = 1 - \gamma$. In the neighborhood of p_{EQ}:
$$f(p) = \begin{cases} f_-(p) = p + \gamma & \text{if} \quad p \in [\gamma, p_{EQ}), \\ f_+(p) = 2p - 1 & \text{if} \quad p \in (p_{EQ}, 1], \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{17}$$
We have $\lim_{p \to p_{EQ}^-} f_-(p) = 1$ with separation from $p_{EQ} = 1 - \gamma$ equal to γ. Moreover, $\lim_{p \to p_{EQ}^+} f_+(p) = 1 - 2\gamma$, which is separated from p_{EQ} by γ.
- $p_{EQ} > 1 - \gamma$. In the neighborhood of p_{EQ}:
$$f(p) = \begin{cases} f_-(p) = 1 & \text{if} \quad p \in [1 - \gamma, p_{EQ}), \\ f_+(p) = 2p - 1 & \text{if} \quad p \in (p_{EQ}, 1], \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{18}$$
The separation of $\lim_{p \to p_{EQ}^-} f_-(p) = 1$ from p_{EQ} is $1 - p_{EQ}$. The separation of $\lim_{p \to p_{EQ}^+} f_+(p) = 2p_{EQ} - 1 = p_{EQ} - (1 - p_{EQ})$ from p_{EQ} is again $1 - p_{EQ}$.

■

C. *Best-of-Group Tournament*

The new population is created from the current one by repeating the following process: First, randomly draw (with replacement) ω individuals from the current population and then allow the fittest member of this subset to parent one offspring. In the best-of-group tournament selection pressure increases with the group size ω.

The replicator equation has the form [5]:
$$f(p) = \begin{cases} 1 - (1 - p)^{\omega} & \text{if} \quad p < p_{EQ}, \\ p^{\omega} & \text{if} \quad p > p_{EQ}, \\ p_{EQ} & \text{if} \quad p = p_{EQ}. \end{cases} \tag{19}$$
Note that when $\omega = 1$, we have a trivial case with replicator function f being the identity function. For larger ω we have the following result for non-trivial equilibrium points $p_{EQ} \in (0, 1)$:

Lemma 5: Consider the simple EGT setting of section III and a two-player symmetric game with two pure strategies and a polymorphic equilibrium $p_{EQ} \in (0, 1)$. The population dynamics is driven by the best-of-group tournament selection with group size ω. Then,

1) if $\omega = 2$, for any sufficiently small $\rho > 0$, p_{EQ} is a $[p_{EQ}(1 - p_{EQ}) - \rho]$-isolating point of discontinuity of f;
2) if $\omega \geq 3$, for any sufficiently small $\rho > 0$, p_{EQ} is a $[\kappa(1 - \kappa^{\omega - 1}) - \rho]$-isolating point of discontinuity of f, where $\kappa = \min\{p_{EQ}, 1 - p_{EQ}\}$.

Proof: The two continuous branches of f, $f_-(p) = 1-(1-p)^\omega$ and $f_+(p) = p^\omega$ are separated at p_{EQ} by $f_-(p_{EQ}) - p_{EQ} = (1-p_{EQ})[1-(1-p_{EQ})^{\omega-1}]$ and $p_{EQ} - f_+(p_{EQ}) = p_{EQ}[1-p_{EQ}^{\omega-1}]$, respectively.

1) Let $\omega = 2$. Then

$$\begin{aligned} f_-(p_{EQ}) - p_{EQ} &= (1-p_{EQ})\ [1-(1-p_{EQ})] \\ &= p_{EQ}\ [1-p_{EQ}] \\ &= p_{EQ} - f_+(p_{EQ}) \end{aligned}$$

2) Let $\omega \geq 3$. We have

$$\begin{aligned} f_-(p_{EQ}) - p_{EQ} &= (1-p_{EQ}) \cdot \\ &\quad [1-(1-p_{EQ})^{\omega-1}] \end{aligned} \tag{20}$$

$$p_{EQ} - f_+(p_{EQ}) = p_{EQ}\ [1-p_{EQ}^{\omega-1}] \tag{21}$$

Consider the difference between the two margins (21) and (21),

$$\begin{aligned} g(p) &= (f_-(p) - p) - (p - f_+(p)) \\ &= 1 - 2p + p^\omega - (1-p)^\omega. \end{aligned} \tag{22}$$

Its second derivative reads

$$g''(p) = \omega(\omega-1)\ [p^{\omega-2} - (1-p)^{\omega-2}]. \tag{23}$$

Note that $g(1/2) = 0$. Furthermore, for $p \in (0, 1/2)$ we have $p < 1-p$, and so $p^{\omega-2} < (1-p)^{\omega-2}$, meaning that $g''(p) < 0$. Since g is concave on $(0, 1/2)$ and $g(0) = g(1/2) = 0$, g is positive on $(0, 1/2)$.

We next show that the graph of g is symmetric around the point $(1/2, 0)$. Consider the transform $\tilde{p} = 1-p$. Then

$$\begin{aligned} g(\tilde{p}) &= 1 - 2\tilde{p} + \tilde{p}^\omega - (1-\tilde{p})^\omega \\ &= 1 - 2(1-p) + (1-p)^\omega - (1-(1-p))^\omega \\ &= -1 + 2p - p^\omega + (1-p)^\omega \\ &= -g(p). \end{aligned}$$

From the symmetry around $(1/2, 0)$ we immediately obtain that g is negative and convex on the interval $(1/2, 1)$. Hence, for non-trivial equilibria $p_{EQ} \in (0, 1/2)$, $\kappa = p_{EQ} = \min\{p_{EQ}, 1-p_{EQ}\}$ and we have

$$f_-(p_{EQ}) - p_{EQ} > p_{EQ} - f_+(p_{EQ}) = \kappa\ [1-\kappa^{\omega-1}].$$

Analogously, for $p_{EQ} \in (1/2, 1)$, we have $\kappa = 1 - p_{EQ} = \min\{p_{EQ}, 1-p_{EQ}\}$ and

$$p_{EQ} - f_+(p_{EQ}) > f_-(p_{EQ}) - p_{EQ} = \kappa\ [1-\kappa^{\omega-1}].$$

■

D. Linear Ranking

In ranking selection, the population is first sorted according to agents' evaluation scores. The individual's reproduction is a linear or exponential function of its ordinal position (rank). Ranking selection is one of the commonly used schemes in *genetic algorithms* [16]. The particular form of linear ranking analyzed in [5] has the higher-scoring strategy receiving a rank of 2 while the other strategy receiving a rank of 1.

The replicator equation for linear ranking is given by [5]:

$$f(p) = \begin{cases} 2p/(p+1), & \text{if } \ p < p_{EQ}, \\ p/(2-p), & \text{if } \ p > p_{EQ}, \\ p_{EQ}, & \text{if } \ p = p_{EQ}. \end{cases} \tag{24}$$

Discontinuity properties of the replicator map (24) can be stated as:

Lemma 6: Consider the simple EGT setting of section III and a two-player symmetric game with two pure strategies and a polymorphic equilibrium $p_{EQ} \in (0, 1)$. The population dynamics is driven by the linear ranking selection. Let $\kappa = \min\{p_{EQ}, 1-p_{EQ}\}$. Then for any sufficiently small $\rho > 0$, p_{EQ} is a $[\frac{\kappa(1-\kappa)}{1+(1-\kappa)} - \rho]$-isolating point of discontinuity of f.

Proof: The two continuous branches of f, $f_-(p) = 2p/(p+1)$ and $f_+(p) = p/(2-p)$ are separated at p_{EQ} by

$$f_-(p_{EQ}) - p_{EQ} = \frac{2p_{EQ}}{p_{EQ}+1} - p_{EQ} = \frac{p_{EQ}(1-p_{EQ})}{1+p_{EQ}} \tag{25}$$

and

$$p_{EQ} - f_+(p_{EQ}) = p_{EQ} - \frac{p_{EQ}}{2-p_{EQ}} = \frac{p_{EQ}(1-p_{EQ})}{1+(1-p_{EQ})}. \tag{26}$$

Consider the difference between the two margins,

$$g(p) = (f_-(p) - p) - (p - f_+(p)) \tag{27}$$

$$= \frac{p(1-p)}{1+p} - \frac{p(1-p)}{1+(1-p)} \tag{28}$$

$$= \frac{p(1-p)(1-2p)}{2+p(1-p)}, \tag{29}$$

with second derivative

$$g''(p) = -4\left[\frac{1}{(1+p)^3} - \frac{1}{(2-p)^3}\right].$$

Note that $g(0) = 0$ and $g(1/2) = 0$. For $p \in (0, 1/2)$ we have $1/(1+p)^3 > 1/(2-p)^3$, meaning that $g''(p) < 0$. Since g is concave on $(0, 1/2)$ and $g(0) = g(1/2) = 0$, g is positive on $(0, 1/2)$.

We next show that the graph g is symmetric around the point $(1/2, 0)$. Consider the transform $\tilde{p} = 1 - p$. Then

$$\begin{aligned} g(\tilde{p}) &= \frac{\tilde{p}(1-\tilde{p})(1-2\tilde{p})}{2+\tilde{p}(1-\tilde{p})} && (30)\\ &= \frac{(1-p)(1-(1-p))(1-2(1-p))}{2+(1-p)(1-(1-p))} && (31)\\ &= -\frac{p(1-p)(1-2p)}{2+p(1-p)} && (32)\\ &= -g(p) && (33)\end{aligned}$$

From the symmetry around $(1/2, 0)$, it follows that g is negative and convex on the interval $(1/2, 1)$. Hence, for $p_{EQ} \in (0, 1/2)$, $\kappa = p_{EQ} = \min\{p_{EQ}, 1 - p_{EQ}\}$ and we have

$$f_-(p_{EQ}) - p_{EQ} > p_{EQ} - f_+(p_{EQ}) = \frac{\kappa(1-\kappa)}{1+(1-\kappa)}. \quad (34)$$

Analogously, for $p_{EQ} \in (1/2, 1)$, $\kappa = 1 - p_{EQ} = \min\{p_{EQ}, 1 - p_{EQ}\}$ and we have

$$p_{EQ} - f_+(p_{EQ}) > f_-(p_{EQ}) - p_{EQ} = \frac{\kappa(1-\kappa)}{1+(1-\kappa)}. \quad (35)$$

■

IV. SHADOWING PROPERTY AND REPLICATOR MAPPINGS

Recall, that we would like to use the shadowing lemma framework for three purposes:

1) *To investigate whether the simulated complex co-evolutionary trajectories under the infinite population assumption represent anything real.* The computer arithmetic operates with finite precision and can only yield pseudo-trajectories that cannot be *a-priori* guaranteed to represent any true trajectory of the given complex system. If the co-evolutionary system has the shadowing property then one can be assured that the observed pseudo-trajectories are shadowed by true ones generated by the underlying system.
2) *To investigate whether constraining to the rational domain for p will not leave theoretical and empirical investigations on $[0, 1]$ groundless.* Transforming an irrational p into a rational population ratio close to p can be viewed as a bounded noise with (arbitrarily) small range size. Shadowing property would mean that possible complex chaotic trajectories on $[0, 1]$ would still be reflected in complex trajectories in the rational domain.
3) *To investigate whether the complex dynamical co-evolutionary patterns under infinite populations indicate complex dynamics in large finite population computer simulations.* For large population size, the effects of finite population size on the strategy ratios p can be considered as bounded noise. Furthermore, the larger the population, the smaller the range size of the noise. In a system with shadowing property, finite population pseudo-trajectories are shadowed by the complex trajectories from the original infinite population model.

Unfortunately, all selection mechanisms considered in Section III are discontinuous at equilibrium. Hence none of those co-evolutionary dynamics possesses the shadowing property. In particular, using Theorem 2 and Lemmas 3–6, we obtain the following main results:

Theorem 7: Consider the simple EGT setting of Section III and a two-player symmetric game with two pure strategies and a polymorphic equilibrium $p_{EQ} \in (0, 1)$. The population dynamics is driven by (μ, λ)-selection with $\lambda/\mu \geq 2$. Then, ϵ-shadowing of the population dynamics is not possible for $0 < \epsilon < \min\{p_{EQ}, 1 - p_{EQ}\}$.

Proof: For $\lambda/\mu \geq 3$, the result is a direct corollary of theorem 2 and lemma 3.

For $\lambda/\mu = 2$, choose any $0 < \epsilon < \min\{p_{EQ}, 1 - p_{EQ}\}$. One can always pick a value of $\rho > 0$ such that $\epsilon < \psi = \min\{p_{EQ}, 1 - p_{EQ}\} - \rho$. By lemma 3 we have that p_{EQ} is a ψ-isolating point of discontinuity of f and hence, by theorem theorem 2, ψ-shadowing of the population dynamics is not possible. Since $\epsilon < \psi$, no ϵ-shadowing of the population dynamics is possible either. ■

Theorem 8: Consider the simple EGT setting of Section III and a two-player symmetric game with two pure strategies and a polymorphic equilibrium $p_{EQ} \in (0, 1)$. The population dynamics is driven by the truncation selection with parameter $\gamma \in (0, 1/2)$. Then, ϵ-shadowing of the population dynamics is not possible for $0 < \epsilon < \min\{p_{EQ}, 1 - p_{EQ}, \gamma\}$.

Theorem 9: Consider the simple EGT setting of Section III and a two-player symmetric game with two pure strategies and a polymorphic equilibrium $p_{EQ} \in (0, 1)$. The population dynamics is driven by the best-of-group tournament selection with group size ω. Then, ϵ-shadowing of the population dynamics is not possible for $0 < \epsilon < \kappa(1 - \kappa^{\omega-1})$, where $\kappa = \min\{p_{EQ}, 1 - p_{EQ}\}$.

Theorem 10: Consider the simple EGT setting of Section III and a two-player symmetric game with two pure strategies and a polymorphic equilibrium $p_{EQ} \in (0, 1)$. The population dynamics is driven by the linear ranking selection. Then, ϵ-shadowing of the population dynamics is not possible for $0 < \epsilon < \frac{\kappa(1-\kappa)}{1+(1-\kappa)}$, where $\kappa = \min\{p_{EQ}, 1 - p_{EQ}\}$.

A. A Hawk-Dove Game Example

As an example, consider a hawk-dove game with the cost of injury twice the gain in winning, $C = 2G$, and a (μ, λ) selection with $\lambda = 2\mu$ (each of μ parents generates two offspring). The replicator equation (6) becomes:

$$f(p) = \begin{cases} 2p & \text{if} \quad p \in [0, 1/2), \\ 1 + 2(p - 1) & \text{if} \quad p \in (1/2, 1], \\ 1/2 & \text{if} \quad p = p_{EQ} = 1/2. \end{cases} \tag{36}$$

The map f acts as left-shift on binary representations of $p \in [0, 1/2) \cup (1/2, 1)$. As such, it can generate a wide variety of dynamical behaviors, dictated by the distribution of digits in the binary expansion $[p]_2$ of the initial condition p. The possible dynamics include periodic orbits of arbitrary periods, a-periodic and 'chaotic' orbits (arising from irrational initial conditions). The difference between the well studied map $r(p) = 2p \bmod 1$ in chaotic dynamics (e.g. [8]) and the map $f(p)$ in (36) - the existence of the additional equilibrium $p_{EQ} = 1/2$ for f - is crucial. Since $r(p)$ is a smooth expanding map on a smooth manifold (unit circle) [17], it has the shadowing property. However, Theorem 7 shows that this is not the case for the system (36).

However, on a reduced set of initial conditions, one can argue that a 'shadowing-like' property holds at least for a particular interpretation of the computer round-off error [18]: If the computer can guarantee only M exact binary digits, then iterative application of f on $0.x_1x_2x_3...$ will lead to the pseudo-trajectory $\mathcal{O}$:

$$0.x_1x_2x_3...x_{M-1}x_M,\ 0.x_2x_3...x_{M-1}x_My_1,$$
$$0.x_3...x_{M-1}x_My_1y_2,\ ...\ 0.y_1y_2...y_M,\ 0.y_2...y_{M+1},\ ...,$$

for some $y_j \in \{0, 1\}$, $j \geq 1$. Consider now the set $\mathcal{E}$ of all pre-images under f of $p_{EQ} = 1/2$,

$$\mathcal{E} = \{p \in [0, 1] \mid f^n(p) = p_{EQ} \text{ for some } n \geq 0\}.$$

The real trajectory of (36) starting in $0.x_2x_3...x_{M-1}x_My_1y_2...$ from the f-invariant set $\Omega = (0, 1) \setminus \mathcal{E}$ will ϵ-shadow the pseudo-trajectory $\mathcal{O}$ with $\epsilon = 2^{-M+1}$.

The set $\mathcal{E}$ contains all $p \in [0, 1]$ whose binary expansion $[p]_2$ contains any finite word over the alphabet $\{0, 1\}$ (including the empty word), followed by digit 1, followed by the right-infinite sequence of 0's:

$$\mathcal{E} = \{p \in [0, 1] \mid [p]_2 = .\{0, 1\}^*1000...\}.$$

The set $\mathcal{E}$ is dense in $[0, 1]$, since for any $p \in [0, 1]$ and arbitrarily small $\epsilon > 0$, there will be a $q \in [0, 1]$, such that $|p - q| < \epsilon$ and $[q]_2$ has an infinite tail of 0's. Analogously, it is easy to show that the set Ω is dense in $[0, 1]$ as well. However, Ω is much larger than $\mathcal{E}$ - in fact, while $\mathcal{E}$ is countable, Ω is uncountable since it contains infinite expansion rational numbers and all irrational numbers in $[0, 1]$. Under f, the set Ω contains seeds for a wide variety of dynamical regimes, including periodic and 'chaotic' orbits.

V. DISCUSSION AND CONCLUSION

In their pioneering work [5], Ficici, Melnik, and Pollack formalized co–evolution within the EGT framework and investigated the impact of various selection mechanisms on the dynamical behavior arising from interactions in an infinite-population of pure strategies. Based on simulation studies with computer generated replicator trajectories, it was argued that simple two-strategy games and various types of replication and selection pressure mechanisms can lead to co-evolutionary dynamics exhibiting a wide variety of complex regimes. While this argument is very interesting, it is rather loose and should be made more rigorous. First, we have rigorously shown that the simulation studies that formed the main basis for arguments about complex co-evolutionary dynamics could be misleading since complex systems without the shadowing property cannot be faithfully simulated on finite precision machines in this manner. Second, the replicator maps in previous studies were constructed under the assumption of infinite populations. When complex dynamics is considered, great care must be taken in studying such replicator dynamics. In infinite populations, only rational proportions of pure strategists are possible. The fact that non-linear selection pressure methods can produce irrational proportions cannot be ignored. In evolutionary systems, the dynamics is usually rather trivial and this is not an issue, but in co-evolution complex dynamics can arise and an analysis is needed to resolve whether such effects can have long lasting consequences. We have shown in this paper that for a range of selection mechanisms this is indeed the case and one has to be careful about such effects. One can no longer simply consider a replicator equation operating on $[0, 1]$, because the proportions may not represent any real infinite population and any approximation by however close rational proportion may be deceiving.

In particular, we have used the framework of shadowing lemma to address the question of relevance of infinite population models for practical finite population regimes when complex co-evolutionary dynamics occurs in infinite populations. We have rigorously shown that lessons learned from (often) easier to analyze infinite population models cannot be transferred to finite (albeit possibly very large) populations. We concentrated on a simple EGT setting of two-player symmetric games with two pure strategies and a polymorphic equilibrium. In this context we proved that for (μ, λ), truncation, sequential tournament, best-of-group tournament and linear ranking selections, the co-evolutionary dynamics do not possess the shadowing property. Analyzing other solution concepts, e.g. the intuitive best-scoring strategy solution concept, is an interesting direction for future work. It would also be interesting to extend the framework to include *finite-time shadowing*, which could open the door for rigorous analysis of complex trajectories over a long times.

Our main theoretical arguments critically rely on trajectories ending up in the fixed point. One can argue that there can be only “very few” of such trajectories in systems we study and so “on average” nothing serious happens. We have two replies to such an argument: **(1)** In a rigorous study, it is enough to find one case where things break down to cast doubt on intuitive beliefs about a studied phenomenon; **(2)** In Section IV.A we actually show an example of a simple system where it is easy to see that the set of initial conditions that can eventually lead to the fixed point is a dense set over $[0, 1]$. So the set of trajectories that can end up in the fixed point is rather generic, even though the dynamics is not contractive.

We believe that the techniques based on shadowing lemma from the field of chaotic dynamics that we introduced here will be of wider use in other similar studies of complex dynamics and finite population effects in evolutionary computation systems. When there is a suspicion that the dynamics can be complex, our framework can provide a bridge between detailed dynamical studies in analytically tractable infinite population settings and the finite population computations performed in practice. In general one would like to avoid ‘uncontrollable’ complex dynamics when running an evolutionary computation system. Tractable infinite population models can provide useful hints about the conditions under which such dynamical regimes can occur and about possible forms they can take. It is very important to establish **(1)** whether infinite population complex dynamic regimes have any relevance for practical finite population computations and if so, **(2)** to what degree the finite population dynamics approximates the infinite population one. The framework introduced in this paper provides a novel approach for addressing these issues.

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