Degree distribution and scaling in the connecting - nearest - neighbors model

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February 3, 2012

Abstract

We present a detailed analysis of the Connecting Nearest Neighbors (CNN) model by Vázquez. We show that the degree distribution follows a power law, but the scaling exponent can vary with the parameter setting. Moreover, the correspondence of the growing version of the Connecting Nearest Neighbors (GCNN) model to the particular random walk model (PRW model) and recursive search model (RS model) is established.

1 Introduction

It is well known, that the structure of growing networks is influenced by the dynamical processes of their creation such as node addition and edge linking [1, 2, 3, 4, 5]. To show this explicitly, several models of dynamically evolving networks have been proposed [3, 4, 6].

Many real networks (such as the Internet, for example) expand in time. The number of nodes added to the system far exceeds the number of vanishing nodes. Such networks are well described by the growing models [4]. The dynamics of the network growth is captured by the integro - differential dynamical equations. Some growing network models are analytically solvable, but often we have to rely on numerical simulations or approximate solutions [1, 3, 4, 5]. Numerical simulations of the growing network usually include the following processes:

- at each time step a new node is added to the network, while bringing m new links
- nodes are labeled according to the time s in which they joined the network
- each link attaches itself to the older nodes with certain probability, for example preferentially.

Barabási and Albert [3] showed, both numerically and analytically, that the preferential node attachment leads to scale free network structure. Preferential node attachment means that the linking probability of the edge from the newly added node to an existing node is proportional to the receiving node degree. Ignoring edge direction, the node degree simply counts the number of edges incident with that node in the network. The notion of "scale free-ness" signifies that there is no dominant scale defining the network, except of its size. Scale free networks have complex structure with degree fluctuations of all sizes, as manifested in the power law degree distribution

$$P(k) \propto k^{-\gamma}, \tag{1}$$

where $P(k)$ denotes the probability of choosing[1] a node with the degree k and $\gamma > 0$ is a scaling exponent depending on the network dynamics. It has been shown analytically, that for the Barabási - Albert model $\gamma_{BA} = 3.0$

[1] w.r.t. uniform probability over nodes in the network

[3]. If the node linking differs from the 'pure' preferential one, γ differs from γ_{BA} [1, 4], or even the network is no longer scale free. For example, if the linking probability is random, degree distribution looses its power law character [4, 5].

There are several real examples of scale free networks created by the self organizing processes. Some of them, such as the Internet at the autonomous system level, social network or language network [7, 10, 11] have even a scale free hierarchical structure. Hierarchy in growing networks has been studied by Ravász and Barabási [12] and also by us [2]. It has been shown, both numerically and analytically, that the hierarchical node organization is reflected in the power law distribution of average clustering coefficients $C(k)$ of nodes with the degree k

$$C(k) \propto k^{-\delta}. \tag{2}$$

Here $\delta > 0$ is a scaling exponent. Scale free hierarchical networks have power law degree and clustering coefficient distributions (1) and (2), respectively.

Pure preferential linking leads to scale free, but not to hierarchical network structure. It has been shown recently [12, 2], that both eqs. (1) and (2) hold, if the process of network growth involves some local dynamics - the first of the new m links brought by a new node s to the system is attached randomly with linking probability $\frac{1}{t}$ (where $t \approx N$, the number of nodes), and the other $m-1$ links attach to the neighbors of the first choice node. The general scale free hierarchical model (SFH model) has been studied numerically [2, 12]. For its simplified version with $m = 2$ (SSFH model) we succeeded in finding scaling exponents γ and δ analytically [2].

A different framework for studying complex networks has been presented by Vázquez in [1] - the network structure is discovered through random walk by several surfers. Such a random walk model can be transformed to our SSFH model [2]. Another model by Vázquez, namely the connecting nearest neighbors (CNN) model (section 2), has been solved analytically using the idea of potential edges and potential degree of the node [1]. Differential equations, describing the model dynamics, were treated using several approximations.

In this paper we present a more accurate analysis of the the CNN model. We show, how the asymptotic degree distribution changes when the line of transcritical bifurcation in the parameter space is crossed. We relate our more detailed analytical solution to the original solution by Vázquez [1]. We also show, that the CNN model corresponds (for certain parameter values) to the other known models described in [1].

The paper has the following organization: In the second chapter we present the CNN model. Chapter 3 is devoted to the qualitative analysis of the nonlinear system of differential equations describing the dynamics of the CNN network. In the fourth chapter the correspondence of the growing version of the CNN model (GCNN model [1]) to the particullar random walk model (PRW model [1]) and the recursive search model (RS model [1]) is discussed. Section 5 concludes the paper by summarizing the main findings of this study.

2 The CNN model

The CNN model has been inspired by social networks [1]. It is assumed that in such networks two sites with a common neighbor are connected with greater probability then two randomly chosen sites. This reflects the fact, that in the social network it is more probable that two people (nodes) know themselves (are connected) if they have a common friend (common neighbor). An analytical understanding of the CNN model has been achieved using the notions of potential edges and potential degree.

The dynamics of the CNN network is defined by the transition rates of link states from node s (each possible link from s to other nodes in the network can be in three possible states - disconnected (d), potential edge (p), or an edge (e)). For each node s the transition rates $\nu_{x\to y}(s)$ $x, y \in \{d, p, e\}$ are defined per link. A potential edge is defined as follows: two nodes are connected by a potential edge, if they are not connected by an edge and they have at least one common neighbor. For each node s in the growing CNN network a degree and a potential degree can therefore be defined.

Consider a network with N nodes. Denote the (real) degree and potential degree of a node s by $k(s, N)$ and $k^*(s, N)$, respectively. The dynamics of the network is described by [1], :

$$\frac{\partial k(s,N)}{\partial N} = \nu_{d\to e}\hat{k}(s,N) + \nu_{p\to e}k^*(s,N) - (\nu_{e\to d} + \nu_{e\to p})k(s,N) \quad (3)$$

$$\frac{\partial k^*(s,N)}{\partial N} = \nu_{d\to p}\hat{k}(s,N) + \nu_{e\to p}k(s,N) - (\nu_{p\to d} + \nu_{p\to e})k^*(s,N) \quad (4)$$

$$\hat{k}(s,N) = N - k(s,N) - k^*(s,N), \quad (5)$$

where $\nu_{d\to e}$, $\nu_{p\to e}$ depend, as will be shown later, only on the network size and $\nu_{d\to p}(s)$, $\nu_{p\to d}(s)$ are given as

$$\nu_{d\to p}(s) = \nu_{d\to e}k(s), \quad \nu_{p\to d}(s) = \nu_{e\to d}k(s).$$

The last two equations hold since if, for example, a node i' is connected to another node i by an edge, potential edges from i' are created to all of the neighbors of the node i.

To simplify eqs. (3–5) several assumptions reflecting to properties of real social networks were introduced [1]:

1. All processes leading to edge deletion are neglected: $\nu_{e\to p} = \nu_{e\to d} = 0$.
2. The transition from a potential edge to an edge has a higher probability of occurrence then the transition from being disconnected to an edge

$$\nu_{p\to e} = \frac{\mu_1}{N}, \qquad \nu_{d\to e} = \frac{\mu_0}{N^2},$$

   where $\mu_1 > 0$ and $\mu_0 > 0$ are constants (for details see [1]).
3. Terms of order $1/N^2$ are omitted.

Under these conditions the system (3–5) turns into

$$\frac{\partial k(s,N)}{\partial N} = \frac{\mu_0}{N} + \frac{\mu_1}{N} k^*(s,N) \tag{6}$$

$$\frac{\partial k^*(s,N)}{\partial N} = \frac{\mu_0 k(s,N)}{N} - \frac{\mu_1}{N} k^*(s,N) \tag{7}$$

with solution (k_0 and k_0^* are positive constants) [1]

$$k(s,N) = k_0 \left(\frac{N}{s}\right)^{\beta} \tag{8}$$

$$k^*(s,N) = k_0^* \left(\frac{N}{s}\right)^{\beta}, \tag{9}$$

where

$$\beta = \frac{\mu_1}{2}\left(-1 + \sqrt{(1 + 4\frac{\mu_0}{\mu_1}}\right). \tag{10}$$

The degree distribution of the system is therefore power law (1) with the exponent $\gamma = 1 + \frac{1}{\beta}$ [5, 6].

3 Analysis of the CNN model

In what follows we suggest a more accurate analysis of the original system (3–5) describing the dynamics of the CNN network. From the three simplifying assumptions described above and employed in [1] we use only the first two. We thus start with the system of differential equations:

$$\frac{\partial k(s,N)}{\partial N} = \frac{\mu_0}{N^2}\left(N - k(s,N) - k^*(s,N)\right) + \frac{\mu_1}{N}k^*(s,N) \quad (11)$$

$$\frac{\partial k^*(s,N)}{\partial N} = \frac{\mu_0}{N^2}k(s,N)\left(N - k(s,N) - k^*(s,N)\right) - \frac{\mu_1}{N}k^*(s,N). \quad (12)$$

Unlike in [1], our aim here is a qualitative analysis of this nonlinear system. We would like to describe asymptotic behavior of solutions of (11-12) in the space of parameters μ_0, μ_1. The goal of the following series of substitutions is to transform the system (11-12) to a more convenient form, which can be approximated by an autonomous nonlinear system.

In the continuum approach [5, 6] N is regarded a continuous variable and s is a node label. First set $N = e^\tau$ and substitute

$$x(s,\tau) = k(s,\tau) + k^*(s,\tau) \quad (13)$$

$$y(s,\tau) = k(s,\tau) + 1, \quad (14)$$

to the system (11–12). We obtain

$$x' = \mu_0(1 - xe^{-\tau})y \quad (15)$$

$$y' = \mu_0(1 - xe^{-\tau}) + \mu_1(x - y + 1), \quad (16)$$

where $x(s,\tau)$ and $y(s,\tau)$ are abreviated as x and y, respectively and we write x' and y' for $\frac{\partial x(s,\tau)}{\partial \tau}$ and $\frac{\partial y(s,\tau)}{\partial \tau}$, respectively.

Using another substitution

$$u = 1 - xe^{-\tau} \quad (17)$$

$$v = ye^{-\tau}, \quad (18)$$

together with abbreviations $u(s,\tau) = u$ and $v(s,\tau) = v$, we again get a new system of differential equations:

$$u' = 1 - u - \mu_0 uv \tag{19}$$
$$v' = \mu_1 - \mu_1 u - \mu_1 v - v + \mu_0 u e^{-\tau} + \mu_1 e^{-\tau}, \tag{20}$$

with obvious interpretation of u' and v'. It is natural to assume that the function $u(s,\tau)$ (17) is bounded[2]. Then for $\tau \to \infty$, the functions $\mu_0 u e^{-\tau}$, $\mu_1 e^{-\tau}$ tend to zero. We approximate the system (19–20) by an autonomous one

$$u' = 1 - u - \mu_0 uv \tag{21}$$
$$v' = \mu_1 - \mu_1 u - \mu_1 v - v. \tag{22}$$

The correctness of the approximation (21-22) can also be shown if we introduce a new function $z = e^{-\tau}$ and rewrite the system (19–20) into a three dimensional one

$$u' = 1 - u - \mu_0 uv \tag{23}$$
$$v' = \mu_1 - \mu_1 u - \mu_1 v - v + \mu_0 uz + \mu_1 z. \tag{24}$$
$$z' = -z. \tag{25}$$

Now, we can see from (23-25) that all of the important dynamics occurs in the (u,v) plane.

The system (21–22) possesses two fixed points

$$A = [1, 0] \tag{26}$$

and

$$B = \left[\frac{\mu_1 + 1}{\mu_0 \mu_1}, \frac{\mu_0 \mu_1 - \mu_1 - 1}{\mu_0 \mu_1 + \mu_0} \right]. \tag{27}$$

Having identified the fixed points, the next step is to characterize their stability type. To that end we linearize (21–22). The linearization matrix M of (21–22) reads

$$M = \begin{pmatrix} -1 - \mu_0 v & -\mu_0 u \\ -\mu_1 & -\mu_1 - 1 \end{pmatrix} \tag{28}$$

[2]As the network size $N = e^{\tau}$ grows, the degrees (both actual and potential) of any node $s \leq N$ cannot grow faster than $c \cdot N$ for some positive constant c.

and at the fixed point A (26) we get

$$M_A = \begin{pmatrix} -1 & -\mu_0 \\ -\mu_1 & -\mu_1 - 1 \end{pmatrix}. \tag{29}$$

Analogously at the fixed point B (27) we get

$$M_B = \begin{pmatrix} -\frac{\mu_0\mu_1}{\mu_1+1} & -\frac{\mu_1+1}{\mu_1} \\ -\mu_1 & -\mu_1 - 1 \end{pmatrix} \tag{30}$$

The types of fixed points A, B depend on eigenvalues of M_A, M_B. General form of eigenvalues of the matrix M (28) is

$$\lambda_{1,2} = \frac{Tr(M)}{2} \pm \sqrt{\left(\frac{Tr(M)}{2}\right)^2 - Det(M)}, \tag{31}$$

where $Tr(M)$ and $Det(M)$ denote trace and determinant of M, respectively. Clearly

$$\frac{Tr(M_A)}{2} = -1 - \frac{\mu_1}{2}, \qquad Det(M_A) = 1 + \mu_1 - \mu_0\mu_1$$

and

$$\frac{Tr(M_B)}{2} = -1 - \mu_1 - \frac{\mu_0\mu_1}{\mu_1 + 1}, \qquad Det(M_B) = -1 - \mu_1 + \mu_0\mu_1.$$

Both traces $Tr(M_A)$ and $Tr(M_B)$ are negative for each $\mu_0 > 0,\ \mu_1 > 0$ (which, of course, is fulfilled for the CNN model), and the determinants have opposite signs, $Det(M_A) = -Det(M_B)$. It therefore follows from (31) that the sign of the expression $1 + \mu_1 - \mu_0\mu_1$ is decisive for the stability type of A and B. In the case that $1 + \mu_1 - \mu_0\mu_1 > 0$, the eigenvalues of M_A are both negative and A is an asymptotically stable fixed point called sink. The eigenvalues of M_B have opposite signs and B is an unstable fixed point (saddle). In the complementary case, $1 + \mu_1 - \mu_0\mu_1 < 0$, A and B change their stability types, now A is a saddle and B is a stable fixed point (sink). The critical case occurs when $1 + \mu_1 - \mu_0\mu_1 = 0$. In that case the fixed points A, B coincide.

This type of stability change is called a transcritical bifurcation (e.g. [8, 9]). In the CNN model the bifurcation occurs when the parameters (μ_0, μ_1) cross the line

$$\mu_0 = 1 + \frac{1}{\mu_1} \tag{32}$$

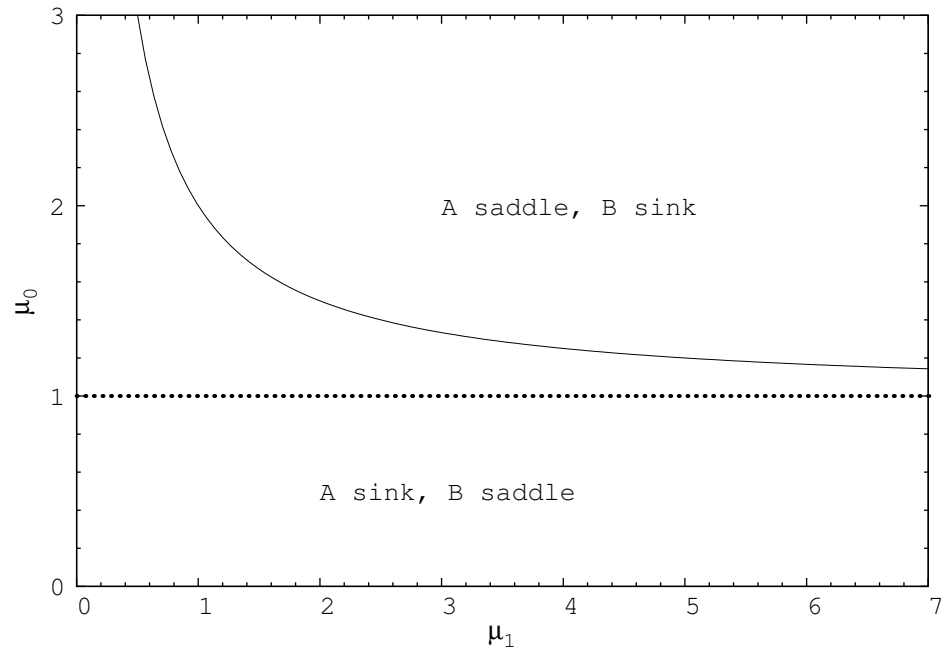

Figure 1: Bifurcation line obtained from (32) in the parameter space of the CNN model delimiting regions of different stability types of fixed points of (11–12). Dotted horizontal line indicates an asymptotic behavior when $\mu_1 \to \infty$.

in the parameter space (see Fig. 1).

Let us now consider both cases in greater detail. In the first case, $1 + \mu_1 - \mu_0\mu_1 > 0$, A is a stable sink and the eigenvalues of M_A are

$$\lambda_{1,2} = -1 - \frac{\mu_1}{2} \pm \sqrt{\mu_0\mu_1 + \frac{\mu_1^2}{4}}. \tag{33}$$

The corresponding eigenvectors of M_A for eigenvalues λ_1, λ_2 read

$$\vec{v}_1 = \begin{pmatrix} -\mu_0 \\ -\frac{\mu_1}{2} + \sqrt{\mu_0\mu_1 + \frac{\mu_1^2}{4}} \end{pmatrix} \tag{34}$$

$$\vec{v}_2 = \begin{pmatrix} -\mu_0 \\ -\frac{\mu_1}{2} - \sqrt{\mu_0\mu_1 + \frac{\mu_1^2}{4}} \end{pmatrix}. \tag{35}$$

The general solution of the linearization which approximates solutions of the system (21–22) and therefore also solutions of (19–20) near the stable fixed point A is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_1 \vec{v}_1 e^{\lambda_1 \tau} + c_2 \vec{v}_2 e^{\lambda_2 \tau}. \tag{36}$$

where c_1, c_2 are constants. Returning to the original variables, neglecting $e^{\lambda_2 \tau}$ terms and using initial condition $k(s,s) = 1$ we get the final approximation

$$k(s,N) = 2\left(\frac{N}{s}\right)^{\beta} - 1 \tag{37}$$

$$k^*(s,N) = 2\left(\frac{\mu_0}{\beta} - 1\right)\left(\frac{N}{s}\right)^{\beta} + 1, \tag{38}$$

where the exponent β is given by

$$\beta = \frac{\mu_1}{2}\left(-1 + \sqrt{1 + \frac{4\mu_0}{\mu_1}}\right). \tag{39}$$

Equation (37) allows us to calculate the average degree as

$$\bar{k}(N) = \frac{1}{N}\sum_{s=1}^{N} k(s,N) = \frac{1}{N}\left[\sum_{s=1}^{N} 2\left(\frac{N}{s}\right)^{\beta} - 1\right]. \tag{40}$$

Approximating the sum in (40) by definite integral, the asymptotic (as $N \to \infty$) average degree reads

$$\bar{k} = \frac{1+\beta}{1-\beta}. \tag{41}$$

There is another way of obtaining the asymptotic average degree $\bar{k}$. From (38),

$$k^*(s,s) = 2\frac{\mu_0}{\beta} - 1 = \sqrt{1 + \frac{4\mu_0}{\mu_1}}. \tag{42}$$

This result has an interesting interpretation. If a node comes and links itself to a node s' by one edge, potential edges are immediately created to all of the neighbors of s'. Therefore $k^*(s,s)$ can be interpreted as an asymptotic average degree. The difference between (41) and (42) is due to the neglected terms in (37), but both calculations lead to the finite average degree $\bar{k}$.

In the other case, when $1+\mu_1-\mu_0\mu_1 < 0$, fixed point B is asymptotically stable and the eigenvalues of M_B are

$$\lambda_{3,4} = -\frac{1+\mu_1}{2} - \frac{\mu_0\mu_1}{2(\mu_1+1)} \pm \sqrt{\left(\frac{1+\mu_1}{2} + \frac{\mu_0\mu_1}{2(\mu_1+1)}\right)^2 - \mu_0\mu_1 + \mu_1 + 1} \tag{43}$$

Clearly $\lambda_4 < \lambda_3 < 0$.
The general solution approximating solutions of the system (19–20) near the stable fixed point B is therefore

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\mu_1+1}{\mu_0\mu_1} \\ \frac{\mu_0\mu_1-\mu_1-1}{\mu_0\mu_1+\mu_0} \end{pmatrix} + c_3\vec{v}_3 e^{\lambda_3\tau} + c_4\vec{v}_4 e^{\lambda_4\tau}, \tag{44}$$

where c_3, c_4 are constants and $\vec{v}_i = (v_{i1}, v_{i2})^T$, $i \in \{3,4\}$ are eigenvectors of M_B. Let us denote $\mu = \frac{\mu_0\mu_1-\mu_1-1}{\mu_0\mu_1+\mu_0}$.

Backward substitutions lead to the solution in the original variables k, k^*:

$$k(s,N) = \mu N - c_3 v_{32} N^{1+\lambda_3} - c_4 v_{42} N^{1+\lambda_4} - 1 \tag{45}$$

$$k^*(s,N) = \frac{\mu}{\mu_1} N - c_3(v_{31}-v_{32}) N^{1+\lambda_3} - c_4(v_{41}-v_{42}) N^{1+\lambda_4} + 1 \tag{46}$$

where coefficients c_3 c_4 are s -dependent.

Since $\lambda_4 < \lambda_3 < 0$, the largest exponent of N is 1 and we can approximate the solution by

$$k(N) = \mu N \tag{47}$$

$$k^*(N) = \frac{\mu}{\mu_1} N \tag{48}$$

For $N \to \infty$,

$$\lim_{N\to\infty} \frac{k(s,N)}{k^*(s,N)} = \mu_1, \tag{49}$$

and μ_1 is thus an asymptotic ratio of the node's degree and its potential degree.

It is possible to find a neater approximation. As $\lambda_4 < -1$, neglecting only terms with $N^{1+\lambda_4}$ in (45) we obtain

$$k(s,N) = \mu N - c_3 v_{32} N^{1+\lambda_3} - 1 \quad (50)$$

$$k^*(s,N) = \frac{\mu}{\mu_1} N - c_3 (v_{31} - v_{32}) N^{1+\lambda_3} + 1 \quad (51)$$

which together with the initial condition $k(s,s) = 1$ leads to the final result

$$k(s,N) = \mu \left(1 - \left(\frac{N}{s}\right)^{\lambda_3}\right) N + 2 \left(\frac{N}{s}\right)^{1+\lambda_3} - 1 \quad (52)$$

$$k^*(s,N) = \frac{\mu}{\mu_1} \left(1 + \left(\frac{N}{s}\right)^{\lambda_3}\right) N - 2 \left(\frac{N}{s}\right)^{1+\lambda_3} - c_3 v_{31} N^{1+\lambda_3} + 1 \quad (53)$$

where

$$v_{31} = 1 + \frac{1}{\mu_1}. \quad (54)$$

Again, from (52) we can derive the average degree:

$$\bar{k}(N) = \frac{1}{N} \sum_{s=1}^{N} k(s,N) = \frac{1}{N} \sum_{s=1}^{N} \mu \left(1 - \left(\frac{N}{s}\right)^{\lambda_3}\right) N + 2 \left(\frac{N}{s}\right)^{1+\lambda_3} - 1. \quad (55)$$

Approximation of the sum in (55) by definite integral and taking the limit $N \to \infty$ leads to

$$\bar{k} = \infty. \quad (56)$$

As before, the same asymptotic result can be obtained from

$$k^*(s,s) = \frac{\mu}{\mu_1} 2s - 1 - c_3 v_{31} s^{1+\lambda_3} \quad (57)$$

and

$$\lim_{s \to \infty} k^*(s,s) = \bar{k}^* = \infty. \quad (58)$$

The asymptotic average degree tends to infinity when the fixed point B is stable.

To conclude our analysis we can state that the CNN model (3-5) exhibits two different types of behavior, deliminated by the bifurcation line (32) in the parameter space (μ_0, μ_1).

For $\mu_0 < 1 + \frac{1}{\mu_1}$ the degree $k(s, N)$ with increasing N grows sublinearly with the exponent β given by (39), where $0 < \beta < 1$. Due to approximations (41), (42) the asymptotic average degree is in this case constant. The degree distribution is a power law with exponent $\gamma = 1 + \frac{1}{\beta}$ [5, 6].

For $\mu_0 > 1 + \frac{1}{\mu_1}$, the growth of the degree $k(s, N)$ is linear (exponent $\beta = 1$) with increasing N. Approximations (56), (58) imply that now the asymptotic average degree tends to infinity. The degree distribution is a power law with the exponent $\gamma = 2$.

Table 1: γ exponent of the degree distribution in a parameter space region where the fixed point A is a sink and B is a saddle. $\mu_0 = 1$, γ_t is calculated with a help of (10, 39) and γ_n is a numerical value.

μ_1	γ_t	γ_n
0.25	3.56	3.49
0.5	3.0	3.03
1.0	2.62	2.66

We supported our analytical results by the numerical simulations. We simulated the model for parameters μ_0, μ_1 in both regions of the parameter space (Fig. 1), using the equations (11, 12). Our network has 100000 nodes. In the first region, where the Vázquez solution holds, we simulated the model for $\mu_0 = 1$ and for the three different values of μ_1. The numerical results are in a good accordance with the theoreticaly predicted values (Tab.1).

Above the bifurcation line the numerical simulations does not converge very well for the parameter values μ_0, μ_1 close to the bifurcation line. We belive, this is partly due to the fact, that the second terms of the equations (45) and (46) influence the convergence. The results for several parameter values are in the (Tab.2). (Fig. 2) shows the convergence of the γ exponent for $\mu_0 = 2, 3$ and growing μ_1.

4 Correspondence of the CNN model to other models

Having obtained analytical solutions for the CNN model, we would like to establish whether some known models are, at least for some parameter settings, identical to the CNN model. We have tested a growing variant of the CNN model (the GCNN model) and the recursive search (RS) model,

Table 2: γ exponent of the degree distribution in a parameter space region where the fixed point A is a saddle and B is a sink. γ_t is a teoretical value of the scaling exponent, and γ_n is a numerical value.

μ_0	μ_1	γ_t	γ_n
2	6	2.0	2.21
2	8	2.0	2.08
3	9	2.0	2.19
3	12	2.0	2.06

which is derived from the PRW model. All of them are described in [1].

4.1 The GCNN model

In [1] Vázquez proposed a growing variant of the CNN model, the GCNN model. Fix some $u \in (0,1)$. The GCNN network evolves by following iteratively performed rules:

1. With probability $1-u$ a new vertex is introduced and a new edge from the new vertex to a randomly selected network node is created.

2. With the probability u one potential edge (selected at random) is converted to an edge.

The author expects that in the regime of large N, the CNN and GCNN models will behave similarly. It has been stated that the evolution rules are consistent with setting $\mu_0 = 1$ and $\mu_1 = \frac{u}{1-u}$ in (6–7):

$$\frac{\partial k(s,N)}{\partial N} = \frac{1}{N} + \left(\frac{u}{1-u}\right)\frac{1}{N}k^*(s,N) \tag{59}$$

$$\frac{\partial k^*(s,N)}{\partial N} = \frac{k(s,N)}{N} - \left(\frac{u}{1-u}\right)\frac{1}{N}k^*(s,N), \tag{60}$$

How to interpret this? The number of nodes N is now proportional to the time of the network development. If the time is measured by coming nodes, it is necessary to rescale k^* by the factor $\frac{u}{1-u}$ which is, in fact, the ratio of transformed potential edges to one node. For our next analysis it is quite correct to deal with the original set of equations (6–7, 59–60), for reasons which are to be made clear later.

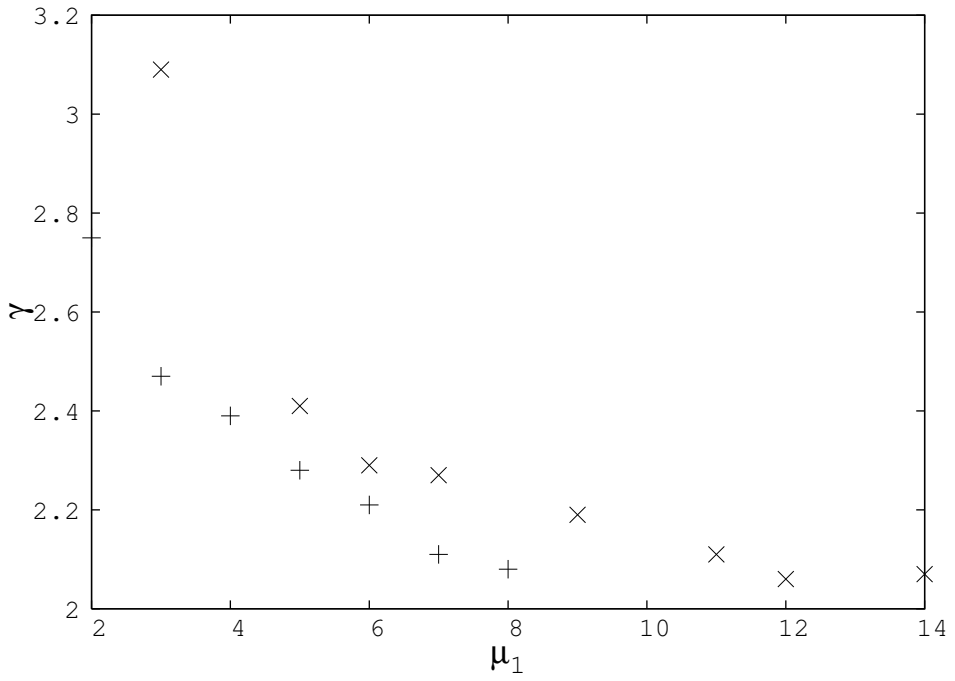

Figure 2: Convergence of the γ exponent to the predicted value $\gamma = 2.0$ in the region above the line of the transcritical bifurcation. $\mu_0 = 3$ for the upper curve (x) and $\mu_0 = 2$ for the lower curve (+). All simulations were done for 100000 nodes.

Having the parameter settings $\mu_0 = 1$ and $\mu_1 = \frac{u}{1-u}$ and taking into account that

$$\gamma = 1 + \frac{1}{\beta}, \tag{61}$$

the scaling exponents β and γ of the GCNN model are

$$\beta = \frac{u}{2(1-u)} \left(-1 + \sqrt{1 + 4\frac{1-u}{u}} \right) \tag{62}$$

and

$$\gamma(u) = 1 + \frac{2(1-u)}{u} \left(-1 + \sqrt{1 + 4\frac{1-u}{u}} \right)^{-1}. \tag{63}$$

The limiting cases, analyzed in [1] are

$$\gamma(0) = \infty \tag{64}$$

and

$$\gamma(1) = 2. \tag{65}$$

Viewing these results in the light of our solution, it is clear that the condition $\mu_0 = 1$ ensures, we are in the region of the parameter space, where the fixed point A is an asymptotically stable sink (Fig. 1). Exponents β and γ are therefore given by (62) and (63). This is also why it is correct to use the original equation set (6, 59), which gives a good result in this part of the parameter space. From (63) we have that $\gamma(0) = \infty$.

There is an alternative argument: If $u = 0$, only the node addition rule works, each new node creating a new edge to a randomly selected node in the network. Such network, growing by the random node addition, has been analyzed in [4, 5]. In such networks the degree distribution decreases exponentially, which corresponds to $\gamma(0) = \infty$.

To analyze the second limiting case, we rewrite (32),

$$\mu_1 = \frac{1}{\mu_0 - 1}. \tag{66}$$

Comparing (66) with $\mu_1 = \frac{u}{1-u}$, it is clear, that for $\frac{1}{\mu_1} = 0$ and $u = 1$ both expressions are identical. That means, that for the limiting case $u = 1$, we are assymptotcally on the bifurcation line (32) in the parameter space, where the solution changes to that described by the equations (47). Here the scaling exponents are $\beta = 1$ and $\gamma = 2$.

Different scaling exponents in the two cases $\gamma(0)$ and $\gamma(1)$ (63) are therefore a natural consequence of the fact, that the solution lies in different regions of the parameter space divided by the line of the transcritical bifurcation (32).

4.2 The GCNN model: $u = \frac{1}{3}$ case

We get another interesting result when $\beta = \frac{1}{2}$ (62). Using (61) leads to $\gamma = 3$. This value of the γ exponent is typical for several well known complex network models, for example the Barabási Albert model (BA model) [3] or the scale free hierarchical model (SFH model) [2]. In [2] we established a simple process of network growth that leads to scale free hierarchical network

structure in which both degree and clustering coefficient distributions are power law (1, 2).

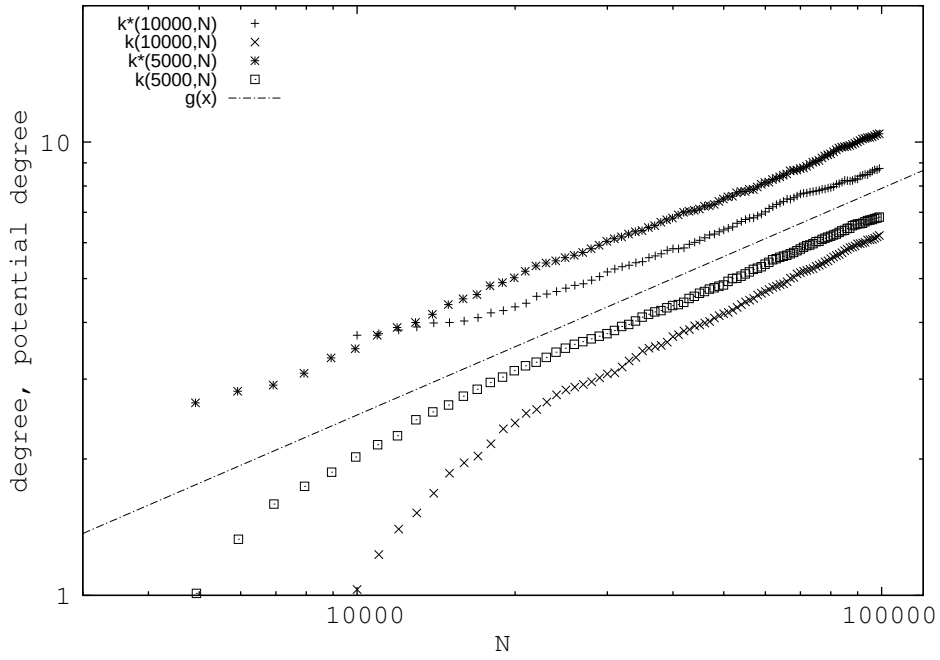

Figure 3: Log-log plot of the degree and potential degree developement with the size of the GCNN network for two different nodes ($s = 5000, 10000$). $k(s, N)$ ($k^*(s, N)$) is a degree (potential degree) of the node s and network size N. The line has a slope $\frac{1}{2}$ (71).

In the GCNN model either new vertex with new edge is added to the system with probability $1 - u$, or, with probability u, some potential edge is chosen and turned into an edge connecting two nodes with a common third node to which the two nodes are connected. A triangle is thus created. Node addition causes creation of several new potential edges in the system. Let us assume that

$$\frac{\partial k^*(s, N)}{\partial N} = \frac{\partial k(s, N)}{\partial N}, \tag{67}$$

holds for the GCNN model for some value of u. Using this assumption, we get from (59)

$$\frac{\partial k^*(s,N)}{\partial N} = \frac{1}{N} + \left(\frac{u}{1-u}\right)\frac{1}{N}k^*(s,N), \tag{68}$$

with the solution

$$k^*(s,N) = \left(\frac{N}{s}\right)^{\frac{u}{1-u}} c - \frac{1-u}{u}. \tag{69}$$

where the exponent $\beta(u) = \frac{u}{1-u}$ and c is a constant. Comparing this value of β with (62) we get that $u = \frac{1}{3}$. That means, that $\beta\left(\frac{1}{3}\right) = \frac{1}{2}$ is a result of the fact, that, as the network grows, both degree and potential degree of nodes change the same way.

Summing (59) and (60) and using the assumption (67) we get

$$2\frac{\partial k(s,N)}{\partial N} = \frac{1}{N}(1 + k(s,N)) \tag{70}$$

with solution (under initial condition $k(s,s) = m$) (Fig.3)

$$k(s,N) = (m+1)\left(\frac{N}{s}\right)^{\frac{1}{2}} - 1 \tag{71}$$

$$k^*(s,N) = (m+1)\left(\frac{N}{s}\right)^{\frac{1}{2}} - 2.$$

To conclude this analysis, we can state that if the special condition $\frac{\partial k^*(s,N)}{\partial N} = \frac{\partial k(s,N)}{\partial N}$ holds in the GCNN model, then $u = \frac{1}{3}$ and the scalings exponents $\beta = \frac{1}{2}$ and $\gamma = 3.0$. It would be interesting to investigate, whether the GCNN model can be transformed to some other known models having degree distribution governed by the exponent $\gamma = 3.0$. We belive, that the scale free hierarchical (SFH) model [2] and the GCNN model are similar, but we leave this to future work.

4.3 Correspondence of the GCNN model to the RS model

Recursive search model (RS model) has been investigated by Vázquez in [1]. It is a variant of his particular random walk model (PRW). General random walk model simulates how the structure of a network is discovered by surfers [1]. The model has been inspired by the manner people obtain information by surfing on the world wide web. Here the web pages are nodes, and hyperlinks are edges.

How does the surfer move? There are two possibilities. Either to jump on a page selected randomly, and then follow a hyperlink, or to jump on another randomly selected page. PRW model is a simpler version of this random walk model. The dynamics of the PRW model consists of adding and walking. The process starts with a single node. Adding means, that the new vertex is created and connected to the one of the existing vertices by an edge. Walking is defined as follows: if an edge is linked to a vertex in the network, the edge is also created to one of its nearest neighbors with probability q_e. When no edge is created, adding rule is applied.

In the RS model the walking rule is slightly changed [1]. Here walking means that if an edge is created to a vertex in the network, then with probability q_e an edge is also created to all of its nearest neighbors. When no edge is created the adding rule is applied with the probability $1 - q_e$.

Vázquez [1] solved the RS model analytically for $q_e = 0$ and $q_e = 1$. In the first case ($q_e = 0$) only adding is present. No link is given to the neighbors of the chosen vertex. We believe that this is the same situation as in the GCNN model with $u = 0$. Here adding creates new potential edges, but none of them is turned to an edge. It has been shown, that the degree distribution in the recursive search model decreases exponentially, which corresponds to the $\gamma(0) = \infty$ (63) case in the GCNN model. And, again, we can say, that the network dynamics is the same as that of the network growing by the random node attachment [4, 3].

In the second limiting case q_e is close to 1 and adding in the RS model is almost not present. But if a node is eventually added, link is created to a randomly chosen node in the network and "potential edges" among the new node and all of the neighbors of chosen node are created and immediately converted to edges. This corresponds to the GCNN model with $u = 1$. Each time step a potential edge is transformed to an edge, with almost no adding, all potential edges are finally transformed into edges. Degree distribution for both models is a power law with scaling exponent $\gamma = 2$.

5 Conclusion

We present a detailed analysis of the CNN network model [1]. The parameter space is divided into two regions in which the CNN model has different behaviour (Fig.1). The degree distribution in both regions is a power law, but the scaling exponent γ is different in different parts of the parameter space. We supported our analytical studies by the numerical simulations. While the results belove the bifurcatin line (Fig.1) closely follow the pre-

dicted values (Table.1), above the line the convergence to the predicted value $\gamma = 2.0$ is worse in the proximity of the bifutcation line (Table.2), (Fig.2). We have nevertheless shown, that the γ exponent converges to the theoretically predicted value.

Our theoretical analysis enabled us to understand the dynamics of the growing variant of the CNN model, the GCNN model, in two limiting cases of $u = 0$ and $u = 1$. The two cases belong to the different regions of the CNN parameter space.

We analysed the GCNN model with $u = \frac{1}{3}$, in which case the scaling exponent γ of the degree distribution is $\gamma = 3.0$. We have found, that this results from the fact, that both the degree and the potential degree of nodes in the GCCN network changes similarly. We speculate that the same holds for the SFH model and that the two models are related when $u = \frac{1}{3}$.

Moreover, we have shown that the RS model and GCNN model in the limiting cases $u = 0, 1$ and $q_e = 0, 1$ have similar dynamics.

Acknowledgemets

Decisive part of this work has been done under the support of the Ramsay - Yunaby research grant at the School of Computer Science, University of Birmingham. This work has been also supported by VEGA grants 1/0476/11 and 2/0019/10. All numerical network simulations were analysed with a help of the Network Workbench tool [13].

We are also gratefull to our collegue Dr Peter Náther for his advices.

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